STRUCTURAL DYNAMICS

BASICS:

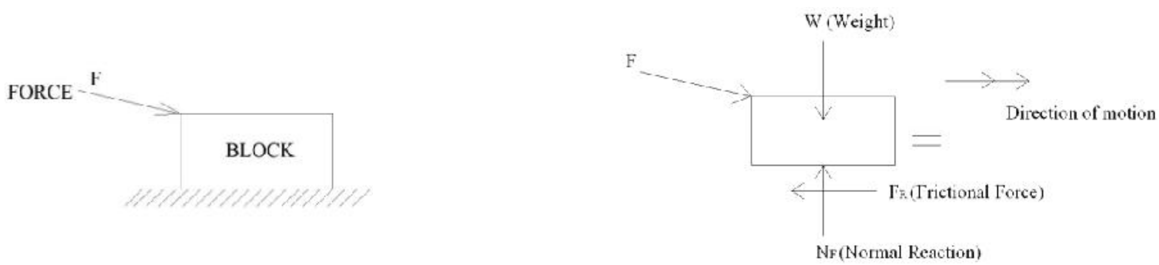
- Real-life structures are subjected to loads which vary with time.
- Except self weight of the structure, all other loads vary with time. In many cases, this variation of the load is small, hence static analysis is sufficient. However, in case of off-shore structures (oil rigs), high rise buildings subjected to lateral loads (wind, earth quake) dynamic effects of the load must be explored for knowing the exact safety and reliability of the structure.

Comparison between static and dynamic analysis:

Static analysis	Dynamic analysis
Loads are constant (magnitude, direction and point of application), hence time invariant.	Loads are varying with time, hence analysis depends on time also.
Static equilibrium is applicable.	Dynamic equilibrium is applicable.
Motion does not occur.	The characteristic of motion in the form of displacement, velocity and acceleration become important parameters.

D'ALEMBERT'S PRINCIPLE:

Consider a block resting as a horizontal surface. Let it be subjected to a force as shown in figure and set to motion. The FBD of the block is as shown.



For the system of forces acting on FBD, we can find a single force called Resultant Force. By Newton's Second Law of Motion, this resultant force must be equal to $R_F = ma$, where m is the mass of the block = $^{W}/_{g}$ and a is acceleration of the block.

To the FBD, if we now add a force, which is equal to R_F in magnitude and opposite in sense, as shown above then this diagram will be in dynamic equilibrium. This force F_i is an imaginary force called as inertial force or reverse effective force.

The principle of adding F_i to FBD is called as D'Alembert's principle.

Some Definitions:

Vibration and oscillation: If motion of the structure is oscillating (pendulum) or reciprocatory along with deformation of the structure, it is termed as VIBRATION. In case there is no deformation which implies only rigid body motion, it is termed as OSCILLATION.

Free vibration: Vibration of a system which is initiated by a force which is subsequently withdrawn. Hence this vibration occurs without the external force.

Forced Vibration: If the external force is also involved during vibration, then it is forced vibration.

Damping: All real life structures, when subjected to vibration resist it. Due to this the amplitude of the vibration gradually, reduces with respect to time. In case of free vibration, the motion is damped out eventually. Damping forces depend on a number of factors and it is very difficult to quantify them. The commonly used representation is viscous damping wherein damping force is expressed as $F_0=C\dot{x}$ where $\dot{x}$ = velocity and C=damping constant.

Degree of Freedom: It is very well known that any mass can have six displacement Components (3 translations and 3 rotations). In most systems, some of these displacements are restrained. The number of possible displacement components is called as Degree of Freedom (DoF). Hence DOF also represents minimum number of coordinate systems required to denote the position of the mass at any instant of time.
An overhead tank is considered as an example. This can be modeled as a cantilever column with concentrated mass at top. If we want axial vibration, then only one coordinate (y) is sufficient. If only the flexural deformation is required then also only one co-ordinate (x) is required. If both are required, then two coordinates are required.
Depending upon the co-ordinates to describe the motion, we have

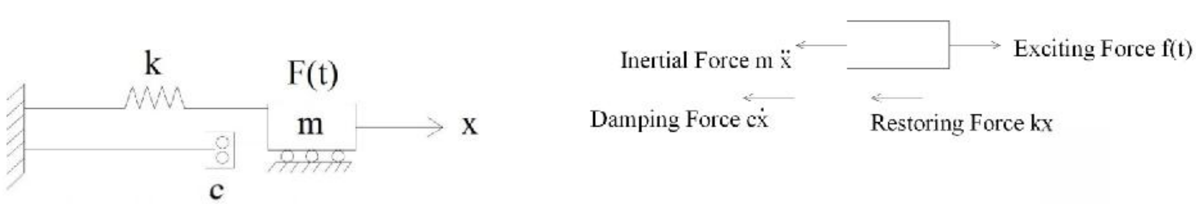
1. Single degree of freedom system (SDoF).
2. Multiple degree of freedom (MDoF).
3. Continuous system.

Free Vibration of SDoF: An SDoF is one which needs only one co-ordinate to describe the motion. The single bay single storey rigid frame is taken as SDoF based on assumptions.

i) Mass of columns are small compared to the mass of the beam. Hence neglected.
ii) Girder is infinitely rigid structure, hence it does not deform and hence the stiffness is provided only by column.
When this frame vibrates due to lateral load in horizontal direction, the force acting are inertial force, (2) damping force, (3) restoring force, and (4) External force.

If the external force is removed after initial disturbance, the free vibration occurs. Further it will be treated as free damped vibration, if damping is present and if damping is not present it is called free undamped vibration.

An SDoF is represented as shown in Figure.



For this system, only one coordinate x is required (translation). Displacement is x, velocity is $\dot{x}\left(\frac{dx}{dt}\right)$ and $\ddot{x}\left(\frac{d^2x}{dt^2}\right)$ is acceleration.

The inertial force hence is $m\ddot{x}$. The damping force is $c\dot{x}$ and spring force is kx.

Using D'Alembert's principle, by dynamic equilibrium.

$$m\ddot{x} + c\dot{x} + kx - f(t) = 0;$$

$$m\ddot{x} + c\dot{x} + kx = f(t); \rightarrow (1)$$

This is 2nd order differential equation. The solution of this equation gives response of an SDoF system.

Free Undamped Vibration: In this case, $f(t)$ is zero and C is zero (because no damping).

Hence (1) is $m\ddot{x} + kx = 0;\ \ddot{x} + \frac{k}{m}x = 0.$

Let $\sqrt{k/m} = p;$ $\quad \ddot{x} + p^2x = 0 \rightarrow (2)$

The solution in the above equation is of the form $x = Ae^{\lambda t} \rightarrow (3)$ Then $\dot{x} = Ae^{\lambda t}(\lambda)$

$\ddot{x} = Ae^{\lambda t}(\lambda^2)$

Therefore, equation (2) is

$\lambda^2 Ae^{\lambda t} + p^2 Ae^{\lambda t} = 0.$

$\lambda^2 + p^2 = 0.$

$\lambda^2 = -p^2 \Longrightarrow \lambda = \pm ip$.

Where $i = \sqrt{-1}$; Hence (3) is $x = A_1e^{ipt} + A_2e^{-ipt}$.

$x = A_1\{\cos pt + i \sin pt\} + A_2\{\cos pt - i \sin pt\} \rightarrow (4)$

On rearranging,

$x = c_1 \cos pt + c_2 \sin pt \rightarrow (5).$

c_1 and c_2 are constants. Since cosine and sine functions are periodic functions, motion defined by x will also be periodic (motion repeats itself after certain interval of time).

T → time period when motion completes one complete rotation.

Since $p = 2\pi \Rightarrow T = \frac{2\pi}{p}, T = 2\pi\sqrt{\frac{m}{k}} \rightarrow (6)$

T is called time period of an undamped free vibration system. Reciprocal of T is the frequency which is nothing but number of times motion repeats itself in one second.

This reciprocal is represented as f and it is natural frequency of the system.

$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \rightarrow (7)$ Hz (cycle/s)

Since $p = \frac{2\pi}{T} = 2\pi f.$

p is called circular frequency or angular frequency of vibration (Rad/s)

Equation (5) is a harmonic motion c_1 and c_2 can be determined from certain initial conditions.

For example: if at $t = 0, x = x_0$ and $\dot{x} = \dot{x}_0$,

From (5) $\dot{x} = c_1p(-\sin pt) + c_2p(\cos pt).$

At $t=0$, $\dot{x} = \dot{x}_0 = c_1 p(-\sin 0) + c_2 p(\cos 0)$.

$\dot{x}_0 = c_2 p \Rightarrow \frac{\dot{x}_0}{p} = c_2$

From (5) At t=0; $x = x_0 = c_1 \cos 0 + c_2 \sin 0$.

$x_0 = c_1 \Longrightarrow c_1 = x_0$.

Therefore (5) is $x = x_0 \cos pt + \frac{\dot{x}_0}{p} \sin pt$ → (9)

Further manipulation is done by multiplying and dividing the terms on RHS of (9) by a factor A. then

$x = A\left\{\frac{x_0}{A} \cos pt + \frac{\dot{x}_0/p}{A} \sin pt\right\}$

A is defined by geometry as $A = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p}\right)^2}$.

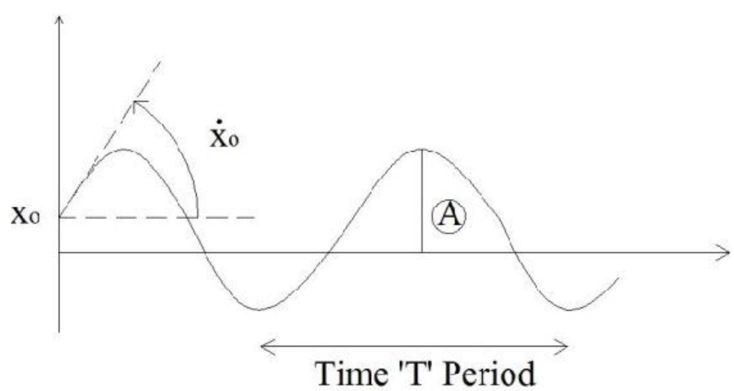
$\sin\beta = \frac{\dot{x}_0/p}{A}$; $\cos\beta = \frac{x_0}{A}$;

Then $x = A\{\cos\beta \cos pt + \sin\beta \sin pt\}$

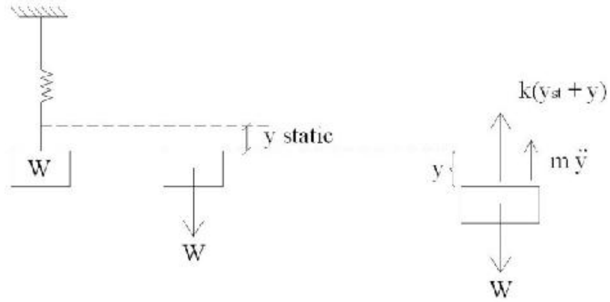
$x = A\{\cos(pt - \beta)\}$→(10).

A in the above equation is the amplitude of motion. Angle β is called phase angle. Equation (10) is harmonic in nature. β which is the phase angle is computed as

$\frac{\sin\beta}{\cos\beta} = \tan\beta = \frac{\dot{x}_0}{p x_0}$.



Problem 1: Weight of 15 N is vertically suspended by a spring of stiffness k=2 N/mm. Determine natural frequency of free vibration of weight.



$y_{static} = {}^{w}/_{k}.$

$m\ddot{y} + k(y_{st} + y) - w = 0.$

${}^{w}/_{g}\,\ddot{y} + k(y_{st} + y) - w = 0.$

${}^{w}/_{g}\,\ddot{y} + k\left({}^{w}/_{k} + y\right) - w = 0.$

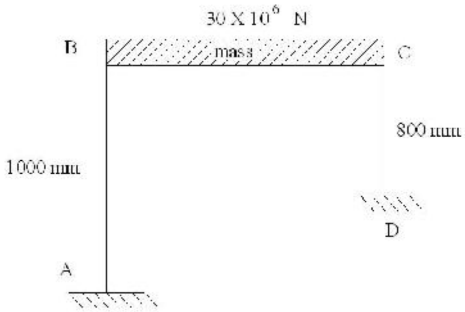
${}^{w}/_{g}\,\ddot{y} + ky = 0. \qquad \sqrt{\frac{k}{m}} = p$

$\ddot{y} + py = 0. \qquad p = \sqrt{\frac{k}{\left(\frac{w}{g}\right)}} = \sqrt{\frac{kg}{w}}.$

$p = \sqrt{\frac{kg}{w}} = \sqrt{\frac{2(9810)}{15}} = 36.166\ Rad/sec.$

$f = \frac{1}{2\pi}p = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 5.76\ Hz\ (cps).$

Problem 2: Calculate the natural angular frequency of the frame shown in figure. Compute also natural period of vibration. If the initial displacement is 25 mm and initial velocity is 25 mm/s what is the amplitude and displacement @t =1s.



In this case, the restoring force in the form of spring force is provided by AB and CD which are columns. The equivalent stiffness is computed on the basis that the spring actions of the two columns are in parallel.

 $k_{eq} = k_{AB} + k_{CD}$.

$= \frac{12\,EI}{1000^3} + \frac{12\,EI}{800^3} = 1063125$ N/mm.

 $Natural\ circular\ frequency = \sqrt{\frac{k_{eq}}{m}}.\ p = \sqrt{\frac{k_{eq}g}{w}}$.

 $p = \sqrt{\frac{1063125\times9810}{30\times10^6}} = 18.645 Rad/sec.$

Natural frequency $f = \frac{1}{2\pi}p = 2.967$ cycles/s.

Natural period = $T = 0.337$ secs.

$x = A\cos(pt - \beta)$.

Given At $t = 0; x_0 = 25\,mm; \dot{x}_0 = 25\,mm/s$

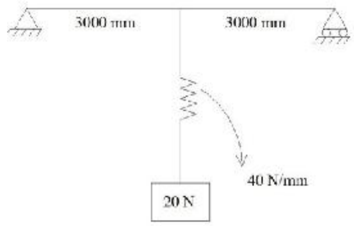
$A = \sqrt{25^2 + \left(\frac{25}{18.645}\right)^2} = 25.03\,mm.$

$\beta = \tan^{-1}\left(\frac{\dot{x}_0/p}{x_0}\right)$ $=\tan^{-1}\left(\frac{25/18.645}{25}\right)$ =0.0535 Radians.

At t=1s,

$x = 25.03\cos(18.645t - 0.0535) = 24.2\,mm.$

Problem 3: Find the natural frequency of the system shown. The mass of the beam is negligible in comparison to the suspended mass. E=2.1 x 10^5 N/mm^2.



The beam has elastic behavior and hence has spring action. Further spring attached to the mass also has a restoring force. The deformation of the mass at the centre is equal to deformation of beam at midspan and that of attached spring.

Central deflection in a beam carrying a single concentrated load is $\delta = \frac{pl^3}{48\,EI}$

 For $\delta = 1$; p is the stiffness which is $k = \frac{48EI}{l^3}$;

Hence the system is considered to contain two springs k_1(beam), k_2(direct spring) which are connected in series.

In a series connection, the equivalent stiffness k_{eq} is

$$1/k_{eq} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$$

$$I = \frac{1}{12}(100)(150)^3 = 28125000\ mm^4.$$

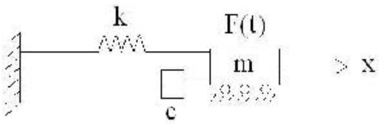
$$EI = 5.90625 \times 10^{12} N - mm^2$$

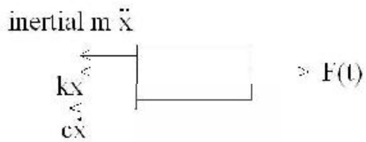
$$k_{eq} = \frac{1312.5(40)}{1312.5+40} = 38.82 \text{ N/mm.}$$

$$p = \sqrt{\frac{k}{m}} = 137.99 \text{ Rad/sec}$$

$$f = \frac{1}{2\pi} p = 21.962 \text{ Hz.}$$

Free damped vibration of SDF system:





In case of free vibration, F(t)=0; $m\ddot{x} + c\dot{x} + kx = 0 \longrightarrow (1)$.

This is a d.e of 2nd order the general solution of which is $x = Ae^{\lambda t} \longrightarrow (2)$.

$\therefore\ \dot{x} = \frac{dx}{dt} = A\lambda e^{\lambda t};\ \ddot{x} = \frac{d^2x}{dt^2} = A\lambda^2 e^{\lambda t};$

Putting, these values in (1) and dividing by m throughout,

$A\lambda^2 e^{\lambda t} + \frac{c}{m}\lambda e^{\lambda t} + \frac{k}{m}Ae^{\lambda t} = 0.$

$Ae^{\lambda t}\left\{\lambda^2 + {}^{\lambda c}/_{m} + {}^{k}/_{m}\right\} = 0$

$\lambda^2 + \lambda\frac{c}{m} + \frac{k}{m} = 0$

Let $\sqrt{\frac{k}{m}} = p$; and $\frac{c}{2m} = n$;

$\lambda^2 + 2n\lambda + p^2 = 0 \longrightarrow (3);$

p → constant for a given system; n → $\frac{c}{2m}$ → c damping constant, m is also a constant.

Hence (3) is quadratic equation in λ and its solution is

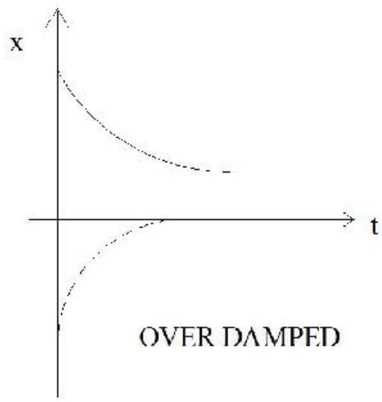
$\lambda_{1,2} = \frac{-2n \pm \sqrt{(2n)^2 - 4p^2}}{2} = -n \pm \frac{\sqrt{4n^2 - 4p^2}}{2}$

$\lambda_{1,2} = -n \pm \sqrt{n^2 - p^2} \longrightarrow (4)$

The net value of $\lambda_{1,2}$ depends on values of n and p. Hence following cases are possible.

Case 1: n>p; in this case, $\lambda_{1,2}$ both are real but negative. The system is said to be over damped. The motion equation is $x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \longrightarrow (5)$

Both λ_1 and λ_2, are negative which implies that motion decays exponentially with time. Hence in this case, there is no periodicity (or harmonic motion) but the damping is so huge, that the system just comes back slowly equilibrium from displaced position.

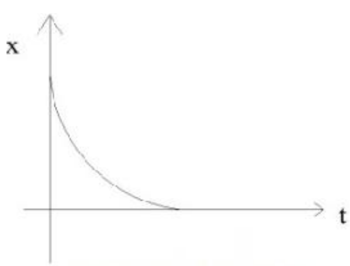


Case 2: n=p;

In this case, in equation (4) the term under the square root vanishes. Hence $\lambda_{1,2}$ are real, negative and equal.

∴ Solution is $x = A_1 e^{\lambda t} + A_2 e^{\lambda t} \longrightarrow (6)$.

Even in this case, there is no oscillatory motion. The damping in the system is enough to bring it back to equilibrium. System is said to be **Critically Damped**.



CRITICAL DAMPED

Since, n = p, $\frac{c}{2m} = p$, c = 2mp.

Since this represents damping constant corresponding to critical system it is expressed as C_c=2mp.

Case 3: n<p;

Since n is less than p, the term under the square root becomes negative. Hence the roots are complex conjugate roots.

 $\lambda_{1,2} = -n \pm i\sqrt{p^2-n^2}$.

 motion equation is $x = A_1 e^{\left\{-n+i\sqrt{p^2-n^2}\right\}t} + A_2 e^{\left\{-n-i\sqrt{p^2-n^2}\right\}t}$

OR $x = A_1 e^{-nt} e^{it\sqrt{p^2-n^2}} + A_2 e^{-nt} e^{-it\sqrt{p^2-n^2}}$

$x = e^{-nt}\left[A_1 e^{it\sqrt{p^2-n^2}} + A_2 e^{-it\sqrt{p^2-n^2}}\right]$

Since, $e^{\pm it\sqrt{p^2-n^2}} = \cos\sqrt{p^2-n^2}\,t \pm i\,\sin\sqrt{p^2-n^2}\,t$

$x = e^{-nt}\left[(A_1+A_2)\cos\sqrt{p^2-n^2}\,t + i(A_1-A_2)\sin\sqrt{p^2-n^2}\,t\right]$

Let $(A_1+A_2) = c_1$; and $i(A_1-A_2) = c_2$

$x = e^{-nt}\left[c_1\cos\sqrt{p^2-n^2}\,t + c_2\sin\sqrt{p^2-n^2}\,t\right] \longrightarrow (7)$

c_1 and c_2 are constants to be calculated from known initial conditions.

At t=0, $x=x_0$ and velocity = $\dot{x_0}$. Using these conditions, c_1 and c_2 can be computed.

At t=0;

$x=x_0$

$x = x_0 = 1[c_1\cos 0 + c_2\sin 0] = 0 \Rightarrow c_1 = 0$

$\dot{x} = e^{-nt}\left[-c_1\sqrt{p^2-n^2}\sin\sqrt{p^2-n^2}\,t + c_2\sqrt{p^2-n^2}\cos\sqrt{p^2-n^2}\,t\right] -$

$ne^{-nt}\left\{c_1\cos\sqrt{p^2-n^2}\,t + c_2\sin\sqrt{p^2-n^2}\,t\right\}$

At t=0; $\dot{x} = \dot{x}_0 = 1\left[x_0(0) + c_2\sqrt{p^2-n^2}\right] - n[x_0(1) + c_2(0)]$

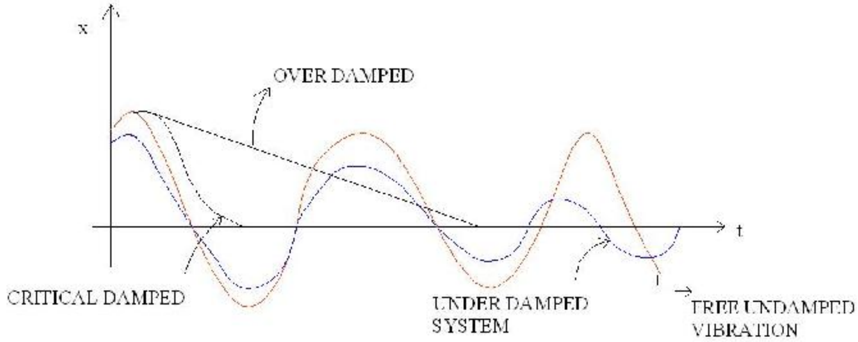
$\dot{x}_0 = c_2\sqrt{p^2-n^2} - nx_0$

$\frac{\dot{x}_0 + nx_0}{\sqrt{p^2-n^2}} = c_2$;

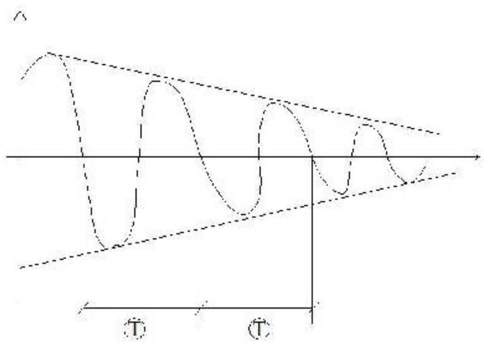
 $x = e^{-nt}\left[x_0 cos\sqrt{p^2-n^2}\,t + \left(\frac{nx_0+\dot{x}_0}{\sqrt{p^2-n^2}}\right) sin\sqrt{p^2-n^2}\,t\right] \longrightarrow (8)$.

Equation (8) describes motion in undamped case.

The amplitude of vibration goes down gradually in an exponential manner.



The variation of the displacement under damped free vibration system is as shown. The amplitude decreases in an exponential manner.



The damped angular frequency P_d is written as $p_d = \sqrt{p^2 - n^2}$.

 Time period = $\frac{2\pi}{p_d} = \frac{2\pi}{\sqrt{p^2-n^2}}$

$$T_d = \frac{2\pi}{p\sqrt{1-(^n/_p)^2}} = \frac{2\pi}{p}\left(\frac{1}{\sqrt{1-(^n/_p)^2}}\right) \rightarrow (9)$$

$\frac{1}{\left(\sqrt{1-(^n/_p)^2}\right)}$ will be greater than one. This implies $T_d > \frac{2\pi}{p}$ (which is natural undamped time period). This means that damped natural time periods is more that of undamped period. However, this increase in time period is very small and hence for all practical purposes, it is assumed that a small viscous damping will not affect time period of vibration.

Problem 1: A platform of weight 18 KN is being supported by four equal columns which are clamped to the foundation. Experimentally, it has been computed that a static force 5KN applied horizontally, to the platform produces a displacement of 2.5 mm. It is estimated that the damping in the structure is of the order of 5% of critical damping. Compute the following:

a) Undamped natural frequency.

b) Damping coefficient.

c) Logarithmic decrement.

d) No of cycles and time required for amplitude of motion to be reduced from an initial value of 2.5 mm to 0.25 mm.

Stiffness is force/unit disp = $\frac{5x10^3}{2.5}$

= 2000 n/mm.

Undamped natural frequency = $\sqrt{\frac{k}{m}}$

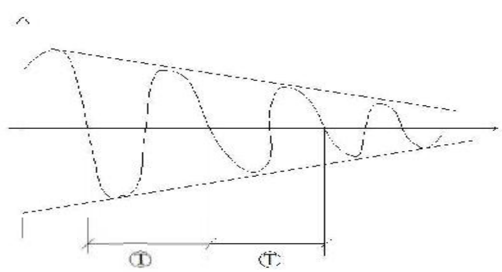
$\sqrt{\frac{2000 \times 9810}{18 \times 10^3}}$ =33.01 Rad/sec.

Critical damping coefficient $c_c = 2\sqrt{km} = 2\sqrt{2000 \times \frac{18\times10^3}{9.81\times10^3}} = 121.16\ NS/mm.$

 actual damping = 5% of C_c = $\frac{5}{100} \times 121.16 = 6.06\ NS/mm.$

Logarithmic decrement:

For a free vibration under damped SDoF system, the osciallatory motion is as shown in the figure. With the increase in time, there is a gradual decrease in amplitude. This delay of amplitude is expressed by using logarithmic decrement 'δ'. δ is the natural logarithm of the ration of any two successive peak amplitude say x_1 and x_2.



 $\delta = ln\left(\frac{x_1}{x_2}\right)$.

At any instant say t_1, $x_1 = e^{-nt_1}\left\{c_1 \cos\sqrt{p^2-n^2}\, t_1 + c_2 \sin\sqrt{p^2-n^2}\, t_1\right\}$.

Next positive amplitude x_2 will occur at time $t = t_2 + T = t_1 + \frac{2\pi}{\sqrt{p^2-n^2}}$

$$\square\ x_2 = e^{-n\left[t_1+\frac{2\pi}{\sqrt{p^2-n^2}}\right]}\left\{c_1 \cos\sqrt{p^2-n^2}\left[t_1+\frac{2\pi}{\sqrt{p^2-n^2}}\right] + c_2 \sin\sqrt{p^2-n^2}\left[t_1+\frac{2\pi}{\sqrt{p^2-n^2}}\right]\right\}.$$

Upon simplification $\frac{x_1}{x_2} = \frac{1}{e^{\frac{-2\pi n}{\sqrt{p^2-n^2}}}}$, taking logarithm on either side $ln\left(\frac{x_1}{x_2}\right) = \delta = \frac{2\pi n}{\sqrt{p^2-n^2}}$

i.e., $\delta = \frac{2\pi^{n}/_{p}}{\sqrt{^{p^2}/_{p^2} - ^{n^2}/_{p^2}}}$

Let $^{n}/_{p} = \xi$, then $\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$.

Since, ξ is very small value we write $\delta = 2\pi\xi$.

Therefore $\xi = \frac{1}{2\pi}\, ln\left(\frac{x_1}{x_2}\right)$

By using this ratio, critical damping is computed.

The ratio between the first amplitude and k^{th} amplitude is $\frac{x_0}{x_k} = \frac{x_0}{x_1}\frac{x_1}{x_2} \dots \dots \frac{x_{k-1}}{x_k}$

Taking logarithm on either side we write $ln\left(\frac{x_0}{x_k}\right) = k\delta$.

In this case, $\delta = 2\pi\xi = 0.314$.

Therefore, $ln\left(\frac{2.5}{0.25}\right) = k(0.314)$

$k = 7.33 \square\ 8\ Cycles$

$p_d = 32.96\ Rad/s$

$T_d = 0.19\ Sec$

Time for 8 Cycles = 1.52 Sec.

Critical damping is defined as the least value of damping for which the system does not osciallate and which the system does not oscillate. When disturbed initially, it simply will return to the equilibrium position after an elapse of time.

Problem 2: A vibrating system consisting of a weight of w=50 N and a spring with stiffness of 4N/mm is viscously damped. The ration of two successive amplitudes is 1:0.85 compute

a) natural frequency (undamped) of the system.

b) logarithmetic decrement

c) damping ratio

d) the damping co.eff and

e) damped natural frequency

undamped natural frequency $p = \sqrt{\frac{k}{m}}$; $m = \frac{50}{9810}$; $K = 4$ *N/mm*

$p = \sqrt{\frac{4}{50/9810}} = 28.01$ *Rad/sec*.

$f = \frac{p}{2\pi} = 4.46$ cycles/sec.

Logarithmic decrement $\delta = ln\left(\frac{x_1}{x_2}\right) = ln\left(\frac{1}{0.85}\right)$; $\delta = 0.162$.

Damping ratio: ξ

$\delta = 2\pi\xi; \xi = \frac{\delta}{2\pi} = \frac{0.162}{2(\pi)} = 0.026.$

Damping co-efficient $=\xi = \frac{c}{2\sqrt{km}}$;

$c = 2\xi\sqrt{km} = 2(0.026)\sqrt{4\left(\frac{50}{9810}\right)} = 0.0074\times10^{-3}$ *NS/mm*

Natural damped frequency p_d.

$p_d = p\sqrt{1-\xi^2} = 28.01\sqrt{1-0.026^2}$; $p_d = 28$ *Rad/Sec*.

Forced vibration of SD_oF system:

Consider a SD_oF system under forced vibration as illustrated.

$m\ddot{x} + c\dot{x} + kx = F(t)$ →(1)

This is a 2nd order ODE for which solution is of the form $x = CF + PI$ →(2).

CF-> Complimentary function which is the solution obtained by taking RHS in equation 1 as zero. Then, the equation obtained is nothing but free vibration case.

PI-> This is the solution obtained which has direct linkage to the forcing function F(t).

F(t) which is the dynamic load varies with time. If F(t) is a harmonic function w.r.t time say $F(t)=F_0 \sin wt$, where F_0->Amplitude, W->Angular frequency, then equation (1) is $m\ddot{x} + c\dot{x} + kx = F_0 \sin wt$ →(3).

Using the terms $n = \frac{c}{2m}$ and $p^2 = k/m$.

$\ddot{x} + 2n\dot{x} + p^2 x = \frac{F_0}{m} \sin wt$ → (4) the solution then is $x = (x)_{CF} + (x)_{PI}$ →(5), where

$x_{(cF)=e^{-nt}\{c_1 \cos\sqrt{p^2-n^2}t + c_2 \sin\sqrt{p^2-n^2}t\}}$ → (6).

The solution as PI is of the form $(x)_{PI} = A\cos wt + B\sin wt$ → (7).

Therefore $(\dot{x})_{PI} = -A\sin wt(w) + B\cos wt(w)$

$(\ddot{x})_{PI} = -A\,w^2 \cos wt - Bw^2 \sin wt$. Substitute these values in equation (4)

$-Aw^2 \cos wt - Bw^2 \sin wt + 2n\{-Aw\sin wt + Bw\cos wt\} + p^2\{A\cos wt + B\sin wt\} = \frac{F_0}{m} \sin wt.$

$\cos wt\{-AW^2 + 2nBw + Ap^2\} + \sin wt\{-Bw^2 - 2nAw + Bp^2\} = \frac{F_0}{m} \sin wt$

Comparing coefficients, $-Aw^2 + 2nBw + Ap^2 = 0$

$-Bw^2 - 2nAw + Bp^2 = \frac{F_0}{m}$

$A(p^2 - w^2) + 2nBw = 0 \quad A = \frac{-2nBw}{(p^2-w^2)}$

$B(p^2 - w^2) - 2nAw = \frac{F_0}{m}$

$B(p^2 - w^2) + 2nw\left\{\frac{2nBw}{p^2-w^2}\right\} = \frac{F_0}{m}$.

$B\left[(p^2 - w^2) + \frac{(2nw)^2}{(p^2-w^2)}\right] = \frac{F_0}{m}.$

$B = \frac{F_0/m(p^2-w^2)}{(p^2-w^2)+(2nw)^2}$

$A = \frac{-2nw}{(p^2-w^2)}\left\{\frac{+\frac{F_0}{m}(p^2-w^2)}{(p^2-w^2)^2+(2nw)^2}\right\}.$

By using trignometrical equations of $\cos\phi = \frac{(p^2-w^2)}{\sqrt{(p^2-w^2)^2+(2nw)^2}}$ and $\sin\phi = \frac{2nw}{\sqrt{(p^2-w^2)^2+(2nw)^2}}$

$$x_{PI} = \left\{\frac{-\frac{F_0}{m}(2nw)}{(p^2-w^2)^2+(2nw)^2}\right\}\cos wt + \left\{\frac{-\frac{F_0}{m}(p^2-w^2)}{(p^2-w^2)^2+(2nw)^2}\right\}\sin wt$$

$$x_{PI} = \frac{\frac{F_0}{m}}{\sqrt{(p^2-w^2)^2+(2nw)^2}}\{\sin wt\ \cos\phi - \cos wt\ \sin\phi\}$$

$$x_{PI} = \frac{\frac{F_0}{m}}{\sqrt{(p^2-w^2)^2+(2nw)^2}}\sin(wt-\phi).$$

$x_{(cF)}$ corresponds to free damped vibration which decays with elapse of time.

$x_{(PI)}$ corresponds to motion relating to the forcing function. Hence it will have the same frequency as the force. The actual motion is superposition of $x_{(cf)}$ and $x_{(PI)}$. Since $x_{(cf)}$ is a decaying component, it is termed as transient vibration, $x_{(PI)}$ is termed as steady state vibration.

Eqn (9) is rewritten as $x = \frac{\frac{F_0}{mp^2}}{\sqrt{\left(1-{}^{w}/_{p}\right)^2+\left(\frac{2n}{p}\cdot\frac{w}{p}\right)^2}}\sin(wt - \phi).$

${}^{w}/_{p} = \eta$ = tuning factor = resonant frequency ratio.

$\frac{\eta}{p} = \xi$ =damping ratio.

$mp^2 = k;$

$\therefore x = \frac{\left({}^{F_0}/_{k}\right)\sin(wt-\phi)}{\sqrt{(1-\eta^2)^3+(2\eta\xi)^2}}$ → (10).

${}^{F_0}/_{k}$ =static displacement $= \delta_{st}$.

$$x = \frac{\delta_{st}}{\sqrt{(1-\eta^2)^3+(2\eta\xi)^2}}\sin(wt - \theta).$$

Max. amplitude of motion hence is $x_{max} = \frac{\delta_{st}}{\sqrt{(1-\eta^2)^2+(2\eta\xi)^2}}$.

The ratio of dynamic displacement (motion) with that of displacement of statics is known as Dynamic Load Factor (DLF).

$$DLF = \frac{x}{\delta_{st}} = \frac{\sin(wt-\phi)}{\sqrt{(1-\eta^2)^2+(2n\xi)^2}}.$$

Maximum value of $DLF = \frac{1}{\sqrt{(1-\eta^2)^2+(2n\xi)^2}}$.

This maximum value of DLF known as magnification factor μ is $\frac{1}{\sqrt{(1-\eta^2)^2+(2n\xi)^2}}$.

In order to get the maximum value of μ, we compute $\frac{d\mu}{d\eta} = \frac{d}{d\eta}\left\{\frac{1}{\sqrt{(1-\eta^2)^2+(2n\xi)^2}}\right\}$

$$\frac{-2\eta\ 2\,(1-\eta^2)+4\xi^2(2\eta)}{-2[(1-\eta^2)^2+(2\eta\xi)^2]^{3/2}} = 0$$

$$-4n(1-\eta^2) + 8n\xi^2 = 0$$

Or $\eta = \sqrt{1-2\xi^2}$ → (11).

ξ being a small quantity, maximum value of μ obtained when frequency of external force is nearly equal to the frequency of the system in free vibration, μ increases rapidly. This condition is called Resonance.

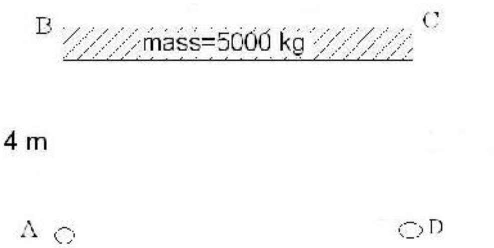
$$\phi = \tan^{-1}\left(\frac{2\eta w}{p^2 - w^2}\right) = \tan^{-1}\left(\frac{2^n/_p{}^w/_p}{(1-(^w/_p)^2)}\right)$$

$$\phi = \tan^{-1}\left(\frac{2\xi\eta}{(1-\eta^2)}\right) \rightarrow (12)$$

Conclusions:

1. The free vibration part is transient and vanishes, forced part persists.
2. With increase of ξ, magnification factor reduces.
3. The magnitude of the maximum value of magnification factor is very sensitive to the value of ξ.
4. Steady state vibration is independent of initial conditions of the system.

Problem 1: A steel rigid frame (one bay one storey) having hinged supports, carries a rotating machine. This escorts a horizontal force at girder level in the form of "50000 sin 11 t" N assuming 4% critical damping, what is steady state amplitude of vibration? I for columns $=1500\times10^{-7}m^4$, $E=21\times10^{10}N/m^2$.



Stiffness for each column = $3EI/l^3$.

Stiffness = $2(3EI/l^3)$

K= 2953125 N/m.

Natural frequency p = $\sqrt{\frac{k}{m}} = 24.3 Rad/s.$

$\xi = 0.04$

$\eta = 0.453$

Therefore $x_{max} = \frac{F_0/k}{\sqrt{(1-\eta^2)^2 + (2\eta\xi)^2}} = 0.0213m = 21.33\ mm.$