STRUCTURAL DYNAMICS

BASICS:

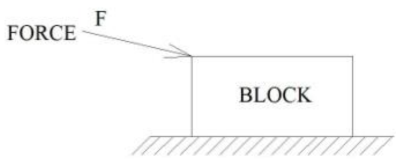
- Real-life structures are subjected to loads which vary with time.
- Except self weight of the structure, all other loads vary with time. In many cases, this variation of the load is small, hence static analysis is sufficient. However, in case of off-shore structures (oil rigs), high rise buildings subjected to lateral loads (wind, earth quake) dynamic effects of the load must be explored for knowing the exact safety and reliability of the structure.

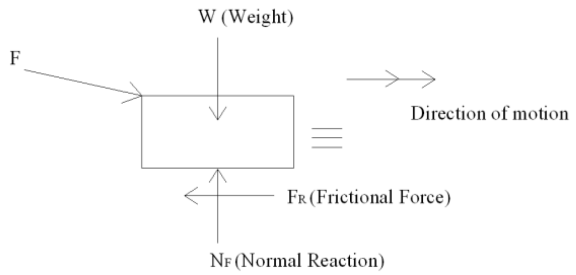
Comparison between static and dynamic analysis:

Static analysis	Dynamic analysis
Loads are constant (magnitude, direction and point of application), hence time invariant.	Loads are varying with time, hence analysis depends on time also.
Static equilibrium is applicable.	Dynamic equilibrium is applicable.
Motion does not occur.	The characteristic of motion in the form of displacement, velocity and acceleration become important parameters.

D'ALEMBERT'S PRINCIPLE:

Consider a block resting as a horizontal surface. Let it be subjected to a force as shown in figure and set to motion. The FBD of the block is as shown.





For the system of forces acting on FBD, we can find a single force called Resultant Force. By Newton's Second Law of Motion, this resultant force must be equal to $R_F = ma$, where m is the mass of the block = $^{w}/_{g}$ and a is acceleration of the block.

To the FBD, if we now add a force, which is equal to R_F in magnitude and opposite in sense, as shown above then this diagram will be in dynamic equilibrium. This force F_i is an imaginary force called as inertial force or reverse effective force.

The principle of adding F_i to FBD is called as D'Alembert's principle.

Some Definitions:

Vibration and oscillation: If motion of the structure is oscillating (pendulum) or reciprocatory along with deformation of the structure, it is termed as VIBRATION. In case there is no deformation which implies only rigid body motion, it is termed as OSCILLATION.

Free vibration: Vibration of a system which is initiated by a force which is subsequently withdrawn. Hence this vibration occurs without the external force.

Forced Vibration: If the external force is also involved during vibration, then it is forced vibration.

Damping: All real life structures, when subjected to vibration resist it. Due to this the amplitude of the vibration gradually, reduces with respect to time. In case of free vibration, the motion is damped out eventually. Damping forces depend on a number of factors and it is very difficult to quantify them. The commonly used representation is viscous damping wherein damping force is expressed as $F_0=C\dot{x}$ where $\dot{x}$ = velocity and C=damping constant.

Degree of Freedom: It is very well known that any mass can have six displacement Components (3 translations and 3 rotations). In most systems, some of these displacements are restrained. The number of possible displacement components is called as Degree of Freedom (DoF). Hence DOF also represents minimum number of coordinate systems required to denote the position of the mass at any instant of time.
An overhead tank is considered as an example. This can be modeled as a cantilever column with concentrated mass at top. If we want axial vibration, then only one coordinate (y) is sufficient. If only the flexural deformation is required then also only one co-ordinate (x) is required. If both are required, then two coordinates are required.
Depending upon the co-ordinates to describe the motion, we have

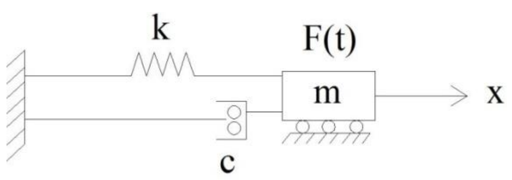
1. Single degree of freedom system (SDoF).
2. Multiple degree of freedom (MDoF).
3. Continuous system.

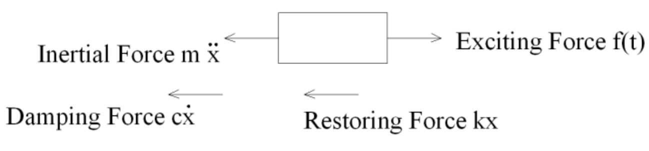
Free Vibration of SDoF: An SDoF is one which needs only one co-ordinate to describe the motion. The single bay single storey rigid frame is taken as SDoF based on assumptions.

i) Mass of columns are small compared to the mass of the beam. Hence neglected.
ii) Girder is infinitely rigid structure, hence it does not deform and hence the stiffness is provided only by column.
When this frame vibrates due to lateral load in horizontal direction, the force acting are inertial force, (2) damping force, (3) restoring force, and (4) External force.

If the external force is removed after initial disturbance, the free vibration occurs. Further it will be treated as free damped vibration, if damping is present and if damping is not present it is called free undamped vibration.

An SDoF is represented as shown in Figure.





For this system, only one coordinate x is required (translation). Displacement is x, velocity is $\dot{x}\ \left(\frac{dx}{dt}\right)$ and $\ddot{x}\left(\frac{d^2x}{dt^2}\right)$ is acceleration.

The inertial force hence is $m\ddot{x}$. The damping force is $c\dot{x}$ and spring force is kx.

Using D'Alembert's principle, by dynamic equilibrium.

$$m\ddot{x} + c\dot{x} + kx - f(t) = 0;$$

$$m\ddot{x} + c\dot{x} + kx = f(t); \rightarrow (1)$$

This is 2nd order differential equation. The solution of this equation gives response of an SDoF system.

Free Undamped Vibration: In this case, $f(t)$ is zero and C is zero (because no damping).

Hence (1) is $m\ddot{x} + kx = 0;\ \ddot{x} + \frac{k}{m}x = 0.$

Let $\sqrt{k/m} = p;$ $\qquad \ddot{x} + p^2x = 0 \rightarrow (2)$

The solution in the above equation is of the form $x = Ae^{\lambda t} \rightarrow (3)$ Then $\dot{x} = Ae^{\lambda t}(\lambda)$

$\ddot{x} = Ae^{\lambda t}(\lambda^2)$

Therefore, equation (2) is

$\lambda^2 Ae^{\lambda t} + p^2 Ae^{\lambda t} = 0.$

$\lambda^2 + p^2 = 0.$

$\lambda^2 = -p^2 \Longrightarrow \lambda = \pm ip$.

Where $i = \sqrt{-1}$; Hence (3) is $x = A_1e^{ipt} + A_2e^{-ipt}$.

$x = A_1\{\cos pt + i\sin pt\} + A_2\{\cos pt - i\sin pt\} \rightarrow (4)$

On rearranging,

$x = c_1\cos pt + c_2\sin pt \rightarrow (5)$.

c_1 and c_2 are constants. Since cosine and sine functions are periodic functions, motion defined by x will also be periodic (motion repeats itself after certain interval of time).

T $\rightarrow$ time period when motion completes one complete rotation.

Since $p = 2\pi \Rightarrow T = \frac{2\pi}{p}, T = 2\pi\sqrt{\frac{m}{k}} \rightarrow (6)$

T is called time period of an undamped free vibration system. Reciprocal of T is the frequency which is nothing but number of times motion repeats itself in one second.

This reciprocal is represented as f and it is natural frequency of the system.

$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \rightarrow (7)$ Hz (cycle/s)

Since $p = \frac{2\pi}{T} = 2\pi f.$

p is called circular frequency or angular frequency of vibration (Rad/s)

Equation (5) is a harmonic motion c_1 and c_2 can be determined from certain initial conditions.

For example: if at $t = 0, x = x_0$ and $\dot{x} = \dot{x}_0$,

From (5) $\dot{x} = c_1p(-\sin pt) + c_2p(\cos pt)$.

Module 1: Introduction to Finite Element Analysis
Lecture 1: Introduction

1.1.1 Introduction

The Finite Element Method (FEM) is a numerical technique to find approximate solutions of partial differential equations. It was originated from the need of solving complex elasticity and structural analysis problems in Civil, Mechanical and Aerospace engineering. In a structural simulation, FEM helps in producing stiffness and strength visualizations. It also helps to minimize materialweight and its cost of the structures. FEM allows for detailed visualization and indicates the distribution of stresses and strains inside the body of a structure. Many of FE software are powerful yet complex tool meant for professional engineers with the training and education necessary to properly interpret the results.

Several modern FEM packages include specific components such as fluid, thermal, electromagnetic and structural working environments. FEM allows entire designs to be constructed, refined and optimized before the design is manufactured. This powerful design tool has significantly improved both the standard of engineering designs and the methodology of the design process in many industrial applications. The use of FEM has significantly decreased the time to take products from concept to the production line. One must take the advantage of the advent of faster generation of personal computers for the analysis and design of engineering product with precision level of accuracy.

1.1.2 Background of Finite Element Analysis

The finite element analysis can be traced back to the work by Alexander Hrennikoff (1941)and Richard Courant(1942). Hrenikoff introduced the framework method, in which a plane elastic medium was represented as collections of bars and beams.These pioneers share one essential characteristic: mesh discretization of a continuous domain into a set of discrete sub-domains, usually called elements.

- In 1950s, solution of large number of simultaneous equations became possible because of the digitalcomputer.
- In 1960, Ray W. Clough first published a paper using term “Finite Element Method”.
- In 1965, First conference on “finite elements” was held.
- In 1967, the first book on the “Finite Element Method” was published by Zienkiewicz and Chung.
- In the late 1960s and early 1970s, the FEM was applied to a wide variety of engineering problems.

- In the 1970s, most commercial FEM software packages (ABAQUS, NASTRAN, ANSYS, etc.) originated.Interactive FE programs on supercomputer lead to rapid growth of CAD systems.
- In the 1980s, algorithm on electromagnetic applications, fluid flow and thermal analysis were developed with the use of FE program.
- Engineers can evaluate ways to control the vibrations and extend the use of flexible, deployablestructures in space using FE and other methods in the 1990s. Trends to solve fully coupled solution of fluid flows with structural interactions, bio-mechanics related problems with a higher level of accuracy were observed in this decade.

With the development of finite element method, together with tremendous increases in computing power and convenience, today it is possible to understand structural behavior with levels of accuracy. This was in fact the beyond of imagination before the computer age.

1.1.3 Numerical Methods

The formulation for structural analysis is generally based on the three fundamental relations: equilibrium, constitutive and compatibility. There are two major approaches to the analysis: Analytical and Numerical. Analytical approach which leads to closed-form solutions is effective in case of simple geometry, boundary conditions, loadings and material properties. However, in reality, such simple cases may not arise. As a result, various numerical methods are evolved for solving such problems which are complex in nature. For numerical approach, the solutions will be approximate when any of these relations are only approximately satisfied. The numerical method depends heavily on the processing power of computers and is more applicable to structures of arbitrary size and complexity. It is common practice to use approximate solutions of differential equations as the basis for structural analysis. This is usually done using numerical approximation techniques. Few numerical methods which are commonly used to solve solid and fluid mechanics problems are given below.

- Finite Difference Method
- Finite Volume Method
- Finite Element Method
- Boundary Element Method
- Meshless Method

The application of finite difference method for engineering problems involves replacing the governing differential equations and the boundary condition by suitable algebraic equations. For example in the analysis of beam bending problem the differential equation is reduced to be solution of algebraic equations written at every nodal point within the beam member. For example, the beam equation can be expressed as:

$$\frac{d^4w}{dx^4} = \frac{q}{EI} \tag{1.1.1}$$

To explain the concept of finite difference method let us consider a displacement function variable namely $w = f(x)$

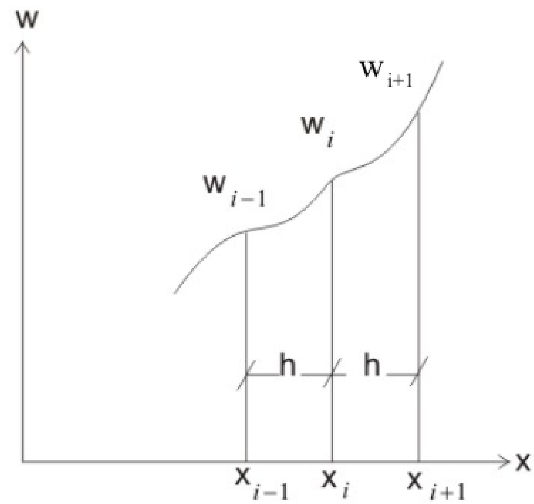


Fig. 1.1.1 Displacement Function

Now, $\Delta w = f(x + \Delta x) - f(x)$

So, $$\frac{dw}{dx} = \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{\Delta w}{\Delta x} = \underset{\Delta x \to 0}{\mathrm{Lt}} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{1}{h}\left(w_{i+1} - w_i\right) \tag{1.1.2}$$

Thus,

$$\frac{d^2w}{dx^2} = \frac{d}{dx}\left[\frac{1}{h}\left(w_{i+1} - w_i\right)\right] = \frac{1}{h^2}\left(w_{i+2} - w_{i+1} - w_{i+1} + w_i\right) = \frac{1}{h^2}\left(w_{i+2} - 2w_{i+1} + w_i\right) \tag{1.1.3}$$

$$\begin{aligned} \frac{d^3w}{dx^3} &= \frac{1}{h^3}\left(w_{i+3} - w_{i+2} - 2w_{i+2} + 2w_{i+1} + w_{i+1} - w_i\right) \\ &= \frac{1}{h^3}\left(w_{i+3} - 3w_{i+2} + 3w_{i+1} - w_i\right) \end{aligned} \tag{1.1.4}$$

$$
\begin{aligned}
\frac{d^4 w}{dx^4} &= \frac{1}{h^4}\left(w_{i+4} - w_{i+3} - 3w_{i+3} + 3w_{i+2} + 3w_{i+2} - 3w_{i+1} - w_{i+1} + w_i\right) \\
&= \frac{1}{h^4}\left(w_{i+4} - 4w_{i+3} + 6w_{i+2} - 4w_{i+1} + w_i\right) \qquad (1.1.5) \\
&= \frac{1}{h^4}\left(w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}\right)
\end{aligned}
$$

Thus, eq. (1.1.1) can be expressed with the help of eq. (1.1.5) and can be written in finite difference form as:

$$
(w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2}) = \frac{q}{EI} h^4 \qquad (1.1.6)
$$

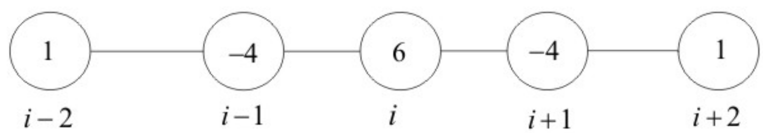


Fig. 1.1.2 Finite difference equation at node i

Thus, the displacement at node *i* of the beam member corresponds to uniformly distributed load can be obtained from eq. (1.1.6) with the help of boundary conditions. It may be interesting to note that, the concept of node is used in the finite difference method. Basically, this method has an array of grid points and is a point wise approximation, whereas, finite element method has an array of small interconnecting sub-regions and is a piece wise approximation.

Each method has noteworthy advantages as well as limitations. However it is possible to solve various problems by finite element method, even with highly complex geometry and loading conditions, with the restriction that there is always some numerical errors. Therefore, effective and reliable use of this method requires a solid understanding of its limitations.

1.1.4 Concepts of Elements and Nodes

Any continuum/domain can be divided into a number of pieces with very small dimensions. These small pieces of finite dimension are called ‘Finite Elements’ (Fig. 1.1.3). A field quantity in each element is allowed to have a simple spatial variation which can be described by polynomial terms. Thus the original domain is considered as an assemblage of number of such small elements. These elements are connected through number of joints which are called ‘Nodes’. While discretizing the structural system, it is assumed that the elements are attached to the adjacent elements only at the nodal points. Each element contains the material and geometrical properties. The material properties inside an element are assumed to be constant. The elements may be 1D elements, 2D elements or 3D elements. The physical object can be modeled by choosing appropriate element such as frame element, plate element, shell element, solid element, etc. All elements are then assembled to obtain the solution of the entire domain/structure under certain loading conditions. Nodes are assigned at a certain density throughout the continuum depending on the anticipated stress levels of a particular domain. Regions which will receive large amounts of stress variation usually have a higher node density than those which experience little or no stress.

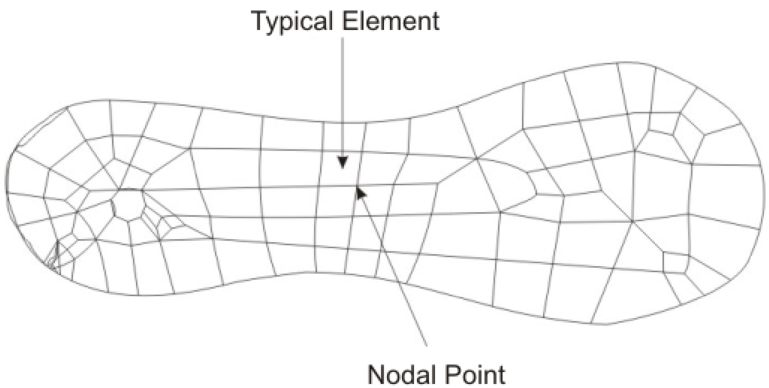


Fig. 1.1.3 Finite element discretization of a domain

1.1.5 Degrees of Freedom

A structure can have infinite number of displacements. Approximation with a reasonable level of accuracy can be achieved by assuming a limited number of displacements. This finite number of displacements is the number of degrees of freedom of the structure. For example, the truss member will undergo only axial deformation. Therefore, the degrees of freedom of a truss member with respect to its own coordinate system will be one at each node. If a two dimension structure is modeled by truss elements, then the deformation with respect to structural coordinate system will be two and therefore degrees of freedom will also become two. The degrees of freedom for various types of element are shown in Fig. 1.1.4 for easy understanding. Here (u, v, w) and $(\theta_x, \theta_y, \theta_z)$ represent displacement and rotation respectively.

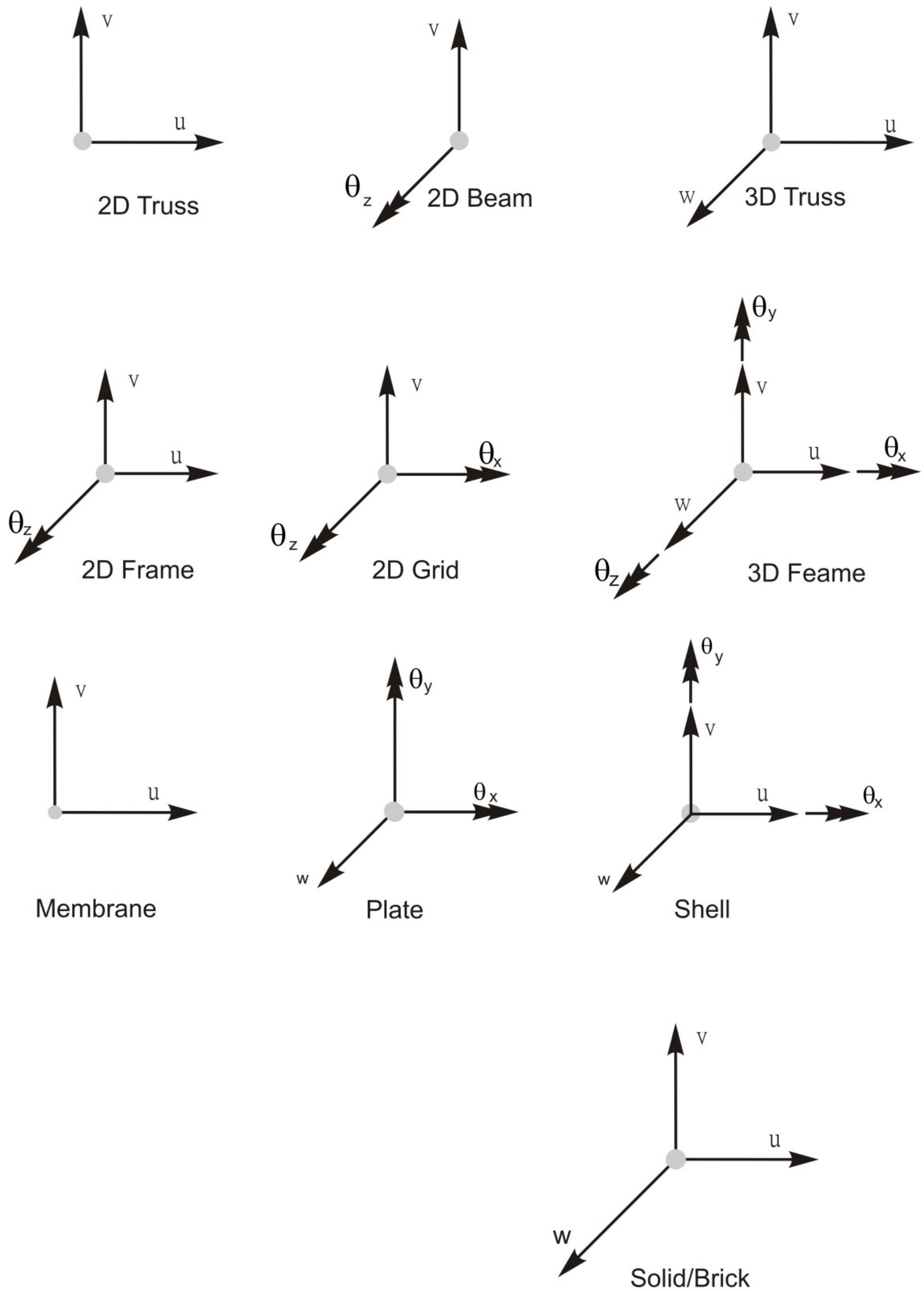


Fig. 1.1.4 Degrees of Freedom for Various Elements